

Neutrino Mass Hierarchies in a Mass Matrix Form Versus its Inverse Form

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Abstract

A neutrino mass matrix model M_ν with $M_\nu^T = M_\nu$ and a model with its inverse matrix form $\widetilde{M}_\nu = m_0^2(M_\nu^*)^{-1}$ can be diagonalized by the same mixing matrix U_ν . It is investigated whether a scenario which provides a matrix model M_ν with normal mass hierarchy can also give a model with an inverted mass hierarchy by considering a model with an inverse form of M_ν .

It is very interesting problem in the current neutrino physics whether neutrino mass hierarchy is normal or inverted. If the observed neutrinos are of Majorana type, models with an inverted hierarchy will be testable by observing the effective neutrino mass $\langle m_{ee} \rangle$ in the near future neutrinoless double beta decay experiments because we expect $\langle m_{ee} \rangle \sim \sqrt{\Delta m_{atm}^2} \simeq 0.05$ eV. On the other hand, if we cannot observe a non-vanishing value of $\langle m_{ee} \rangle$ up to ~ 0.01 eV, as far as the hierarchy is concerned, its test will be hopeless in the near future experiments. For the prediction whether the neutrino mass hierarchy is normal or inverted, current models are half and half. However, in this paper, we notice that, in a model with a mass matrix form M_ν ($M_\nu^T = M_\nu$) which can provide reasonable masses $m_{\nu i}$ ($i = 1, 2, 3$) and mixing U_ν , a mass matrix with an inverse form $\widetilde{M}_\nu = m_0^2(M_\nu^*)^{-1}$ can give the same mixing U_ν as in the original model M_ν . This means a possibility that a model with a normal mass hierarchy can also give an inverted hierarchy. Then, most current models will be able to predict an inverted hierarchy, i.e. most models have a possibility $\langle m_{ee} \rangle \sim 0.05$ eV. This will encourage experimental physicists who have a plan to measure $\langle m_{ee} \rangle$ with the order of 0.05 eV.

Such a scenario with “dual” hierarchy is the following case: Let us consider an effective neutrino mass matrix M_ν on a flavor basis on which the charged lepton mass matrix M_e is diagonal (or nearly diagonal). The mass matrix form M_ν is invariant under a flavor basis transformation T , i.e.

$$T^T M_\nu T = M_\nu. \quad (1)$$

We may consider such operators T more than two. In other words, we consider that the mass matrix M_ν has been derived from the constraints (1). Since we know a no-go theory [1] for flavor symmetries, we consider that the mass matrix M_ν is an effective type, e.g. the form M_ν is described in terms of Higgs scalars more than two, or it is of a Froggatt-Nielsen type [2]. (Therefore, the matrices T can operate only to neutrino sector independently of the charged lepton sector.) We also define a neutrino mass matrix \widetilde{M}_ν as

$$\widetilde{M}_\nu = m_0^2(M_\nu^*)^{-1} + m_0 \xi_0 \mathbf{1}. \quad (2)$$

From the inverse of Eq.(1), we obtain

$$T^T(M_\nu^*)^{-1}T = (M_\nu^*)^{-1}, \quad (3)$$

so that

$$T^T \widetilde{M}_\nu T = m_0^2 (M_\nu^*)^{-1} + m_0 \xi_0 T^T T. \quad (4)$$

Therefore, for a case that the matrices T are orthogonal (e.g. an exchange operator between ν_μ and ν_τ in the 2-3 symmetry [3], and so on), we obtain

$$T^T \widetilde{M}_\nu T = \widetilde{M}_\nu, \quad (5)$$

i.e. the mass matrix form \widetilde{M}_ν is also allowed under the same symmetry. (Note that for operators T with $T^T T \neq \mathbf{1}$ the matrix \widetilde{M}_ν is not invariant under the symmetry. For such a general case with $T^T T \neq \mathbf{1}$, the invariance (5) holds only when $\xi_0 = 0$. On such a case, we will comment later.)

On the other hand, the neutrino mass matrix M_ν is diagonalized as

$$U_\nu^T M_\nu U_\nu = D_\nu \equiv \text{diag}(m_1, m_2, m_3). \quad (6)$$

Therefore, the matrix \widetilde{M}_ν is also diagonalized as

$$U_\nu^T \widetilde{M}_\nu U_\nu = m_0^2 (D_\nu^*)^{-1} + m_0 \xi_0 U_\nu^T U_\nu = \text{diag}(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3), \quad (7)$$

where we have assumed that M_ν is real matrix, so that $U_\nu^T U_\nu = \mathbf{1}$. Therefore, we can obtain the same neutrino mixing matrix U_ν for \widetilde{M}_ν , too. The mass eigenvalues \tilde{m}_i of \widetilde{M}_ν are given by

$$\tilde{m}_i = m_0 \left(\frac{m_0}{m_i} + \xi_0 \right). \quad (8)$$

When we define the ratio R

$$R \equiv \frac{m_2^2 - m_1^2}{m_3^2 - m_2^2}, \quad (9)$$

correspondingly to the observed ratio $R_{obs} = \Delta m_{solar}^2 / \Delta m_{atm}^2$, we obtain

$$\tilde{R} \equiv \frac{\tilde{m}_2^2 - \tilde{m}_1^2}{\tilde{m}_3^2 - \tilde{m}_2^2} = \left(\frac{m_3}{m_1} \right)^2 \frac{1 + 2\xi_0 m_2 m_1 / (m_2 + m_1) m_0}{1 + 2\xi_0 m_3 m_2 / (m_3 + m_2) m_0} R. \quad (10)$$

Therefore, we can, in general, fit the value of \tilde{R} to the observed value by adjusting the free parameter ξ_0/m_0 suitably. In fact, we can obtain $\tilde{R} = \mp R$ by choosing the parameter ξ_0/m_0 as

$$\frac{\xi_0}{m_0} = \frac{(m_3^2 \pm m_1^2)(m_3 + m_2)(m_2 + m_1)}{2m_3 m_2 m_1 [m_3(m_3 + m_2) \pm m_1(m_2 + m_1)]}. \quad (11)$$

We have an interest in whether a model \widetilde{M}_ν gives an inverted mass hierarchy (IH) or not when the original model M_ν gives a normal mass hierarchy (NH). For convenience, we define that $m_2^2 > m_1^2$ (so that the model gives $\tan^2 \theta_{\text{sol}} \sim 0.5$), so that a case with NH gives $R > 0$. Since the mixing matrix U_ν is identical both for M_ν and \widetilde{M}_ν , the masses \widetilde{m}_i have also to satisfy the relation $\widetilde{m}_2^2 > \widetilde{m}_1^2$ because \widetilde{M}_ν will also give $\tan^2 \theta_{12} \sim 0.5$. Therefore, \widetilde{R} takes $\widetilde{R} > 0$ or $\widetilde{R} < 0$ according as NH or IH in \widetilde{M}_ν as well as those in M_ν . In order to see whether the model \widetilde{M}_ν gives NH or IH, we calculate $\widetilde{x} \equiv \widetilde{m}_2/\widetilde{m}_3$ correspondingly to $x \equiv m_2/m_3$. If we see $\widetilde{x}^2 \ll 1$ ($\widetilde{x}^2 \gg 1$) for $x^2 \ll 1$, the case describes a model with NH (IH). We can describe the value \widetilde{x} as a function of x by using the constraint (11):

$$\widetilde{x} = -\frac{\frac{m_2-m_1}{m_2+m_1} + \frac{m_2-m_1}{m_3} + \frac{m_2 m_1}{m_3^2}}{1 + \frac{m_2^2+m_1^2}{(m_2+m_1)m_3} - \frac{m_2 m_1}{m_3^2}}, \quad (12)$$

for $\widetilde{R}/R = +1$, and

$$\widetilde{x} = -\frac{\frac{m_2-m_1}{m_2+m_1} \left(1 + \frac{m_2}{m_3}\right) + \frac{m_2}{m_3^2} \left(1 - \frac{m_2}{m_3}\right)}{1 + \frac{(m_2-m_1)m_2}{(m_2+m_1)m_3} + \frac{(m_2-m_1)(2m_2+m_1)m_1}{(m_2+m_1)m_3^2} - \frac{m_2 m_1^2}{m_3^3}}, \quad (13)$$

for $\widetilde{R}/R = +1$. The value of \widetilde{x} is highly sensitive to the explicit value of x , so that we illustrate the behavior of \widetilde{x} versus x in Fig.1. Note that since $m_1^2 = (1+R)m_2^2 - Rm_3^2$ from the definition (9), we obtain a constraint

$$x^2 > \frac{R}{1+R}, \quad (14)$$

for $R > 0$ (NH), e.g. $|x| > 0.165$ for the observed value $R = 0.028$ [4, 5]. (For a case $R < 0$ (IH), the relation (14) is always satisfied, so that such a constraint does not appear.) In Fig.1 (a), (b) and (c), we have shown only cases in which the original model M_ν gives NH ($x^2 < 1$ and $R > 0$) and the inverse matrix model \widetilde{M}_ν gives NH ($\widetilde{x}^2 < 1$ and $\widetilde{R}/R = +1$) or IH ($\widetilde{x}^2 > 1$ and $\widetilde{R}/R = -1$) under the constraints $\Delta m_{21}^2 > 0$ and $\Delta \widetilde{m}_{21}^2 > 0$. As seen in Fig.1 (b), we find that we can always choose the value of $\widetilde{x} = \widetilde{m}_2/\widetilde{m}_3$ which can give IH with the same value $|\widetilde{R}|$ as R by adding a suitable shift-term $m_0 \xi_0 \mathbf{1}$ to the inverse matrix $m_0^2 (M_\nu^*)^{-1}$. Since the case (c) in Fig.1 gives $1/\widetilde{x} > 0.3$ ($1/\widetilde{x} \sim 0.3$ at $x \sim 0.2$), the case may be called as a case of a nearly degenerated hierarchy (DH) rather than a case of IH.

In this paper, we have interested in a case of NH→IH. An inverse case, IH→NH, can be guessed from the case of NH→IH [Fig.1 (b) and (c)]. For reference, we illustrate the remaining case, IH→IH, in Fig.1 (d). As seen in Fig.1, the cases NH→IH and IH→NH take place in the case with $m_1/m_2 < 0$, and the cases NH→NH and IH→IH take place in the case with $m_1/m_2 > 0$.

So far, we have investigated a model under the constraints $T^T T = \mathbf{1}$ and $U_\nu^T U_\nu = \mathbf{1}$. However, the existence of such constraints narrows applicable cases of our statement to models. Finally, we would like to comment on a model with $\xi_0 = 0$. Then, a mass matrix M_ν can always

provide the inverse matrix model $\widetilde{M}_\nu \equiv m_0^2(M_\nu^*)^{-1}$ which satisfies the same flavor symmetry $T^T \widetilde{M}_\nu T = \widetilde{M}_\nu$ and which has the same mixing matrix U_ν independently of whether $T^T T \neq \mathbf{1}$ and $U_\nu^T U_\nu \neq \mathbf{1}$ or not. The problem is whether such an inverse matrix \widetilde{M}_ν can also give a reasonable mass ratio \widetilde{R} or not. As seen in Eq.(10), such a case gives

$$\widetilde{R} = \left(\frac{m_3}{m_1} \right)^2 R, \quad (15)$$

so that it seems to be ruled out because of the factor $(m_3/m_1)^2$ in Eq.(15). It is true as far as we want a model with an exact relation $|\widetilde{R}| = |R|$. However, the value of $\Delta m_{21}^2 \equiv m_2^2 - m_1^2$ is highly sensitive to the renormalization group equation (RGE) effects and some unknown loop corrections in a beyond-standard model. [The magnitudes of the effects are dependent on the model (the so-called $\tan\beta$ in SUSY model, and so on), so that the effects are not always conclusive.] For example, for a model of M_ν in which a predicted value of Δm_{21}^2 at a higher energy scale is too small value compared with the observed value of Δm_{solar}^2 and a predicted value of $(m_3/m_1)^2$ is not so large, it can be possible that the inverse matrix model \widetilde{M}_ν gives a reasonable value of \widetilde{R} . Therefore, we may expect that some of models M_ν can provide a reasonable masses and mixing for the inverse matrix model \widetilde{M}_ν , too.

In conclusion, we have investigated whether a scenario which provides a matrix model M_ν with normal mass hierarchy can also give a model with an inverted mass hierarchy by considering a model with an inverse form of M_ν . If the model M_ν is derived under a flavor symmetry $T^T M_\nu T = M_\nu$ with $T^T T = \mathbf{1}$, the inverse matrix model \widetilde{M}_ν which is defined by (2) can also satisfy the same symmetry as $T^T \widetilde{M}_\nu T = \widetilde{M}_\nu$, so that the scenario can give both model M_ν and \widetilde{M}_ν . Then, we have find that if a model M_ν with normal hierarchy gives $m_1/m_2 < 0$, the inverse matrix model \widetilde{M}_ν can give a reasonable value of \widetilde{R} with an inverted hierarchy. Even for a case without the constraints $T^T T = \mathbf{1}$ and $U_\nu^T U_\nu = \mathbf{1}$ (so that $\xi_0 = 0$ in Eq.(2)), we still have a possibility that the inverse matrix model \widetilde{M}_ν gives a reasonable value of \widetilde{R} although we cannot give the exact relation $|\widetilde{R}| = |R|$. We may expect the observation of $\langle m_{ee} \rangle \sim 0.05$ eV in the near future neutrinoless double beta decay experiments by considering their inverse matrix models for a considerable number of current models.

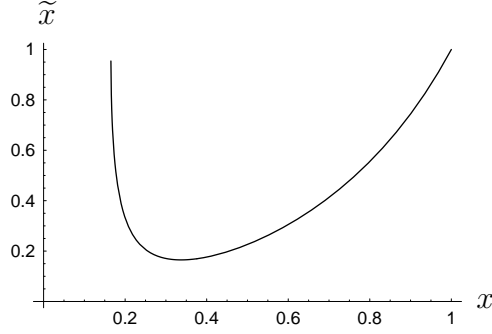
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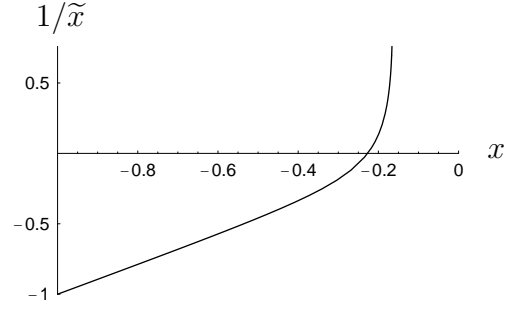
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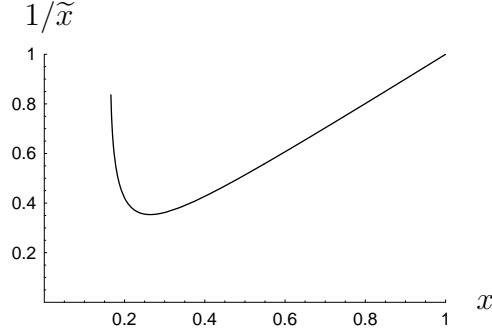
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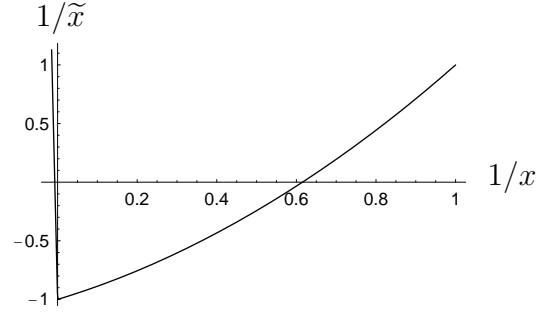
(a) Case NH \rightarrow NH



(b) Case NH \rightarrow IH



(c) Case NH \rightarrow IH



(d) Case IH \rightarrow IH

Fig. 1 Behaviors of $\tilde{x} = \tilde{m}_2/\tilde{m}_3$ versus $x = m_2/m_3$ under the requirement of $|\tilde{R}| = |R|$: Input value $R = +0.028$ has been used as a NH case in M_ν . (a) a case of NH \rightarrow NH in $m_1/m_3 > 0$ and $m_2/m_3 > 0$; (b) a case of NH \rightarrow IH in $m_1/m_3 > 0$ and $m_2/m_3 < 0$; (c) a case of NH \rightarrow IH in $m_1/m_3 < 0$ and $m_2/m_3 > 0$; (d) a case of IH \rightarrow IH in $m_1/m_3 > 0$ and $m_2/m_3 > 0$. In the cases (b) and (c), NH \rightarrow IH, the vertical axis is illustrated as $1/\tilde{x}$ instead of \tilde{x} . Also, in the case (d), IH \rightarrow IH, the horizontal and vertical axes are illustrated by $1/x$ and $1/\tilde{x}$, respectively.